Pearson Edexcel

# Mark Scheme (Results)

## January 2023

Pearson Edexcel International Advanced Level
In Further Pure Mathematics F3 (WFM03)
Paper 01

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **1(a)** | $\frac{dy}{dx} = 3\arcsin 2x + 3x\frac{1}{\sqrt{1-(2x)^2}} \times 2$ <br> $\left(= 3\arcsin 2x + \frac{6x}{\sqrt{1-4x^2}}\right)$ | M1: Obtains <br> $p\arcsin qx + \frac{rx}{\sqrt{1-(sx)^2}}$ **or** <br> $p\arcsin qx + \frac{rx}{\sqrt{1-tx^2}}$ <br> $p, q, r, s, t > 0$ <br><br> A1: Correct derivative. <br> Allow unsimplified and isw. <br><br> Allow $\sin^{-1}$ and condone "arsin" but "arsinh" or "arcsinh" is M0 | M1 A1 |
| **(b)** | $x = \frac{1}{4} \Rightarrow \frac{dy}{dx} = \frac{\pi}{2} + \sqrt{3}$ | $\frac{\pi}{2} + \sqrt{3}$ only but allow $\frac{1}{2}\pi$ or $0.5\pi$ . <br> Terms as a sum in either order. <br> Allow $a = \frac{1}{2}, b = \sqrt{3}$ <br> Isw following a correct answer. | B1**dep** |
| | This is a "Hence" question so this mark can only be awarded following full marks in part (a) | | |
| | | | **Total 3** |

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| --- | --- | --- | --- |
| **2(a)** | $x=-\frac{4}{3}$ | $x=-\frac{4}{3}$ or any equivalent **equation**.<br>Allow $x=\pm\frac{4}{3}$ | B1 |
| | | | **(1)** |
| **(b)(i)**<br>**Way 1** | $\frac{a}{e}=\frac{4}{3}$<br>$b^2=a^2\left(e^2-1\right)\Rightarrow 5=a^2\left(\frac{9a^2}{16}-1\right)$ | Uses $\frac{a}{e}=\pm\frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in *a*. Condone replacing $b^2$ with 25 if equation is otherwise correct | M1 |
| | $9a^4-16a^2-80=0$<br>$\Rightarrow\left(9a^2+20\right)\left(a^2-4\right)=0\Rightarrow a^2=...$ | Solves a 3TQ in $a^2$ (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of $a^2$ or *a* correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g.,<br>"$\left(9a^2+20\right)\left(a^2-4\right)=0\Rightarrow a=4$"<br>**Requires previous M mark.** | **d**M1 |
| | $a=2$ | Not $a=\pm 2$ unless negative rejected | A1 |
| | | | **(3)** |
| **Way 2** | $\frac{a}{e}=\frac{4}{3}$<br>$b^2=a^2\left(e^2-1\right)\Rightarrow 5=\left(\frac{4e}{3}\right)^2\left(e^2-1\right)$ | Uses $\frac{a}{e}=\pm\frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in *e*. Condone replacing $b^2$ with 25 if equation is otherwise correct | M1 |
| | $16e^4-16e^2-45=0$<br>$\Rightarrow\left(4e^2-9\right)\left(4e^2+5\right)=0\Rightarrow e^2=...$ | Solves a 3TQ in $e^2$ (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of $e^2$ or *e* correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g.,<br>"$\left(4e^2-9\right)\left(4e^2+5\right)=0\Rightarrow e=\frac{9}{4}$"<br>**Requires previous M mark.** | **d**M1 |
| | $\left(e=\frac{3}{2}\Rightarrow\right)\quad a=2$ | Not $a=\pm 2$ unless negative rejected but condone sight of "$e=\pm\frac{3}{2}$" or "$e=-\frac{3}{2}$" | A1 |
| | | | **(3)** |

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|---|---|---|---|
| **2(b)(ii)** | $e=\frac{3}{2}\Rightarrow ae=\frac{3}{2}\times 2$ or $ae=\frac{3a^2}{4}=\frac{3}{4}\times 4$ or $ae=c=\sqrt{a^2+b^2}=\sqrt{2^2+5}$ | Uses a correct method to obtain a numerical expression for *ae* oe with their values of *a*, *e*, $a^2$, $b^2$ etc. however obtained. Condone use of a negative *e* or *a* | M1 |
| | Foci are $(\pm 3,0)$ | Both correct foci as coordinates | A1 |
| | Allow "$\frac{a}{e}=\frac{4}{3}\Rightarrow a=4,\ e=3$" to access the last M mark only in (b) for $(\pm 12,0)$ provided the values of both *a* and *e* are clearly seen beforehand | | **(2)** |
| | | | **Total 6** |
| | Note that it is possible to answer (ii) before (i) – e.g.,<br>Let foci be $(\pm c,\ 0)$<br>$a^2e^2=c^2=b^2+a^2=5+a^2$ and<br>$\frac{a}{e}=\frac{a^2}{ae}=\frac{a^2}{c}=\frac{4}{3}\Rightarrow a^2=\frac{4}{3}c$<br>$\Rightarrow c^2=5+\frac{4}{3}c$ ( (i) M1: Uses correct formulae to form an equation in *c* – condone $b^2$ replaced with 25 as with main scheme)<br>$\Rightarrow 3c^2-4c-15=0\Rightarrow(3c+5)(c-3)=0\Rightarrow c=3$<br>( (i) **d**M1: Solves 3TQ to find positive real root)<br>$\Rightarrow(\pm 3,\ 0)$ ((i)A1: Correct foci as coordinates)<br>$a=\sqrt{\frac{4}{3}\times 3}$ ( (ii) M1: Correct method for *a*)<br>*a* = 2 ( (ii) A1: Correct value) | | |

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| **3**<br><br>**Way 1**<br><br>**Converts to sinh and cosh** | $4\tanh x - \text{sech}\, x = 1$<br>$4\frac{\sinh x}{\cosh x} - \frac{1}{\cosh x} = 1$<br>$4\sinh x - 1 - \cosh x = 0$<br>$4\frac{e^x - e^{-x}}{2} - 1 - \frac{e^x + e^{-x}}{2} = 0$ | Replaces **one** hyperbolic function with its correct exponential equivalent. Allow for correct replacement of just e.g., $\sinh x$ after using $\tanh x = \frac{\sinh x}{\cosh x}$ May follow errors but do not allow any further marks if the original equation was reduced to one in a single hyperbolic function. | M1 |
| | $3e^{2x} - 2e^x - 5 = 0$ | M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in $e^x$<br>A1: Correct 3TQ | M1 A1 |
| | $e^x = \frac{2 \pm \sqrt{4+60}}{6} \left( \Rightarrow \frac{2+8}{6} = \frac{5}{3} \right)$ | M1: Solves 3TQ (or 2TQ with no constant) in $e^x$. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation.<br>A1: Any correct unsimplified expression for $e^x$ that includes the positive root. Must be exact | M1 A1 |
| | $x = \ln\frac{5}{3}$ | $\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k$ = ...<br>No unrejected extra solutions | A1 |
| | | | **Total 6** |
| **Way 2**<br><br>**Straight to $e^x$** | $4\frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{2}{e^x + e^{-x}} = 1$ | Replaces **one** hyperbolic function with its correct exponential equivalent | M1 |
| | $3e^{2x} - 2e^x - 5 = 0$ | M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in $e^x$<br>A1: Correct 3TQ | M1 A1 |
| | $e^x = \frac{2 \pm \sqrt{4+60}}{6} \left( \Rightarrow \frac{2+8}{6} = \frac{5}{3} \right)$ | M1: Solves 3TQ (or 2TQ with no constant) in $e^x$. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation.<br>A1: Any correct unsimplified expression for $e^x$ that includes the positive root. Must be exact | M1 A1 |
| | $x = \ln\frac{5}{3}$ | $\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k$ = ...<br>No unrejected extra solutions | A1 |
| | | | **Total 6** |
| | In Ways 1 & 2, if they form an equation which is not a quadratic in $e^x$ they must achieve the correct exact root of $\frac{5}{3}$ to access the middle four marks | | |

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| **3** <br><br> **Way 3a** <br><br> **Squaring (sinh)** | $4\sinh x-1=\cosh x$ <br> $16\sinh^2 x-8\sinh x+1=\cosh^2 x$ <br> $16\sinh^2 x-8\sinh x+1=1+\sinh^2 x$ | Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sinh *x* | M1 |
| | $15\sinh^2 x-8\sinh x=0$ | M1: Obtains a 2TQ with no constant or 3TQ in sinh *x* <br> A1: Correct 2TQ | M1 A1 |
| | $\sinh x=\frac{8}{15}$ | Solves 2TQ (with no constant) or 3TQ in sinh *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. | M1 |
| | $x=\text{arsinh}\frac{8}{15}=\ln\left(\frac{8}{15}+\sqrt{\left(\frac{8}{15}\right)^2+1}\right)$ <br> or $15e^{2x}-16e^x-15=0\Rightarrow$ <br> $e^x=\frac{16\pm\sqrt{256+900}}{30}$ | A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for $e^x$ if they revert to exponentials). Must be exact | A1 |
| | $x=\ln\frac{5}{3}$ | $\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... <br> No unrejected extra solutions | A1 |
| | | | **Total 6** |
| **Way 3b** <br><br> **Squaring (sech)** | $4\tanh x=1+\text{sech}\, x$ <br> $16\tanh^2 x=1+2\text{sech}\, x+\text{sech}^2 x$ <br> $16\left(1-\text{sech}^2 x\right)=1+2\text{sech}\, x+\text{sech}^2 x$ | Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sech *x* | M1 |
| | $17\text{sech}^2 x+2\text{sech}\, x-15=0$ | M1: Obtains a 2TQ (with no constant) or 3TQ in sech *x* <br> A1: Correct 3TQ | M1 A1 |
| | $\left(17\text{sech}\, x-15\right)(\text{sech}\, x+1)=0$ <br> $\text{sech}\, x=\frac{15}{17}$ | Solves 2TQ with no constant or 3TQ in sech *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. | M1 |
| | $x=\text{arcosh}\frac{17}{15}=\ln\left(\frac{17}{15}+\sqrt{\left(\frac{17}{15}\right)^2-1}\right)$ <br> or $15e^{2x}-34e^x+15=0\Rightarrow$ <br> $e^x=\frac{34\pm\sqrt{1156-900}}{30}$ | A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for $e^x$ if they revert to exponentials). Must be exact | A1 |
| | $x=\ln\frac{5}{3}$ | $\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... <br> No unrejected extra solutions | A1 |
| | | | **Total 6** |

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| **3** **Way 3c** **Squaring (tanh)** | $4\tanh x - 1 = \text{sech}\, x$ <br> $16\tanh^2 x - 8\tanh x + 1 = \text{sech}^2 x$ <br> $16\tanh^2 x - 8\tanh x + 1 = 1 - \tanh^2 x$ | Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in tanh *x* | M1 |
| | $17\tanh^2 x - 8\tanh x = 0$ | M1: Obtains a 2TQ with no constant or 3TQ in tanh *x* <br> A1: Correct 2TQ | M1 A1 |
| | $\tanh x = \frac{8}{17}$ | Solves 2TQ with no constant or 3TQ in tanh *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. | M1 |
| | $x = \text{artanh}\frac{8}{17} = \frac{1}{2}\ln\left(\frac{1+\frac{8}{17}}{1-\frac{8}{17}}\right)$ <br> or $9e^{2x} - 25 = 0 \Rightarrow$ <br> $e^x = \frac{5}{3}$ | A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for $e^x$ if they revert to exponentials). Must be exact | A1 |
| | $x = \ln\frac{5}{3}$ | $\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... <br> No unrejected extra solutions | A1 |
| | | | **Total 6** |
| **Way 3d** **Squaring (cosh)** | $4\sinh x = 1 + \cosh x$ <br> $16\sinh^2 x = 1 + 2\cosh x + \cosh^2 x$ <br> $16\cosh^2 x - 16 = 1 + 2\cosh x + \cosh^2 x$ | Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in cosh *x* | M1 |
| | $15\cosh^2 x - 2\cosh x - 17 = 0$ | M1: Obtains a 2TQ with no constant or 3TQ in cosh *x* <br> A1: Correct 3TQ | M1 A1 |
| | $(15\cosh x - 17)(\cosh x + 1) = 0$ <br> $\cosh x = \frac{17}{15}$ | Solves 2TQ (with no constant) or 3TQ in cosh *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. | M1 |
| | $x = \text{arcosh}\frac{17}{15} = \ln\left(\frac{17}{15} + \sqrt{\left(\frac{17}{15}\right)^2 - 1}\right)$ <br> or $15e^{2x} - 34e^x + 15 = 0 \Rightarrow$ <br> $e^x = \frac{34 \pm \sqrt{1156 - 900}}{30}$ | A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for $e^x$ if they revert to exponentials). Must be exact | A1 |
| | $x = \ln\frac{5}{3}$ | $\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... <br> No unrejected extra solutions | A1 |
| | | | **Total 6** |

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| **4(a)** | $\int \frac{1}{\sqrt{9x^2+16}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{x^2+\frac{16}{9}}}\,dx$<br>$= \frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)$ or $\frac{1}{3}\text{arsinh}\left(\frac{x}{\frac{4}{3}}\right) \quad (+c)$<br>or $\frac{1}{3}\ln\left(x+\sqrt{x^2+\left(\frac{4}{3}\right)^2}\right) \quad (+c)$ | M1: Obtains<br>$p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$<br>or $t\ln\left(ux+\sqrt{vx^2+w}\right)$<br>$p, q, r, s, t, u, v, w > 0$<br>A1: Any correct expression. Could be unsimplified and isw. The "+$c$" is not required. Allow $\sinh^{-1}$ and condone "arcsinh".<br>"arcsin" or "arsin" is M0 | M1 A1 |
| | | | **(2)** |
| **(b)** | $\int_{-2}^{2} \frac{1}{\sqrt{9x^2+16}}\,dx$<br>$= \left[\frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{-2}^{2}$ or $\left[\frac{2}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{0}^{2}$<br>$= \frac{1}{3}\text{arsinh}\left(\frac{3\times 2}{4}\right) - \frac{1}{3}\text{arsinh}\left(\frac{3\times -2}{4}\right)$ or $\frac{2}{3}\text{arsinh}\left(\frac{3}{2}\right)$<br>OR<br>$\left[\frac{1}{3}\ln\left(x+\sqrt{x^2+\frac{16}{9}}\right)\right]_{-2}^{2}$<br>$= \frac{1}{3}\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right) - \frac{1}{3}\ln\left(-2+\sqrt{(-2)^2+\frac{16}{9}}\right)$<br>or $\frac{2}{3}\left(\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right) - \ln\left(0+\sqrt{0^2+\frac{16}{9}}\right)\right)$ | Substitutes the limits 2 and -2 into an expression of the form<br>$p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$<br>or $t\ln\left(ux+\sqrt{vx^2+w}\right)$<br>$p, q, r, s, t, u, v, w > 0$<br>and subtracts either way round or obtains an expression for $2[...]_0^{\pm 2}$<br>The expression does not have to be consistent with their answer to (a).<br>No rounded decimals unless exact values recovered.<br>Any f(0) = 0 can be implied by omission.<br>Condone poor bracketing. | M1 |
| | $\frac{1}{3}\ln\left(\frac{11}{2}+\frac{3\sqrt{13}}{2}\right)$ or $\frac{1}{3}\ln\frac{11+3\sqrt{13}}{2}$<br>or $\frac{2}{3}\ln\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)$ or $\frac{2}{3}\ln\frac{3+\sqrt{13}}{2}$ | **d**M1: Obtains an expression of the form<br>$a\ln\left(b+c\sqrt{13}\right)$ *or* $a\ln\left(\frac{d+e\sqrt{13}}{f}\right)$<br>where $a, b, c, d, e, f$ are exact and $> 0$.<br>Condone poor bracketing.<br>**Requires previous M mark.**<br>A1: Any correct equivalent in an appropriate form (fractions may not be in simplest form) with correct bracketing if necessary and isw. **Must come from correct work**.<br>Allow e.g., $a=\frac{2}{3}, b=\frac{3}{2}, c=\frac{1}{2}$ | **d**M1 A1 |
| | For information the decimal answer is 0.7965038115 | | **(3)** |
| | | | **Total 5** |

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| **5(a)** | $\begin{vmatrix} a & a & 1 \\ -a & 4 & 0 \\ 4 & a & 5 \end{vmatrix}$ <br> $= a(4\times5-0)-a(-5a-0)+1\left(-a^2-(4\times4)\right)$ | Uses a correct method for det **A** (implied by two correct parts) to obtain an expression in *a* | M1 |
| | $\Rightarrow 20a+5a^2-a^2-16=0$ <br> $\Rightarrow a^2+5a-4=0$ <br> $\Rightarrow a=\dfrac{-5+\sqrt{41}}{2}$ | Correct exact value oe <br> Condone $\dfrac{-5\pm\sqrt{41}}{2}$ | A1 |
| | | | **(2)** |
| **(b)(i)** <br> **Way 1** <br> $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert$ | $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert = \begin{vmatrix} a-\lambda & a & 1 \\ -a & 4-\lambda & 0 \\ 4 & a & 5-\lambda \end{vmatrix}$ <br> $=(a-\lambda)(4-\lambda)(5-\lambda)-a\times-a(5-\lambda)+\left(-a^2-4(4-\lambda)\right)$ <br> or $\lvert\mathbf{A}-2\mathbf{I}\rvert = \begin{vmatrix} a-2 & a & 1 \\ -a & 2 & 0 \\ 4 & a & 3 \end{vmatrix}$ <br> $=6(a-2)-a\times-3a+\left(-a^2-8\right)$ | Obtains an expression for $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert$ in terms of $a$ and $\lambda$ or just *a* if $\lambda$ is replaced by 2. <br> If method unclear insist on 2 out of 3 correct parts. <br> May multiply along any row/column. <br> Sarrus leads to the same expressions shown (or the expressions all multiplied by -1 if "=0"). | M1 |
| | $\lambda=2\Rightarrow(a-2)\times2\times3+3a^2-a^2-8=0$ <br> $2a^2+6a-20=0 \;\Rightarrow\; a^2+3a-10=0$ <br> $\Rightarrow\;(a-2)(a+5)=0\;\Rightarrow a=...$ | Following use of $\lambda=2$, forms and solves a 3TQ in *a*. Apply usual rules. If no working they must obtain one correct solution for their 3TQ which could be complex. Could be implied. <br> **Requires previous M mark.** | **d**M1 |
| | $(a>0\;\Rightarrow)\,a=2$ | Correct value of *a* from correct work. <br> If –5 is offered imply its rejection if 2 alone is used in (ii) | A1 |
| | If *a* = 2 is arrived at fortuitously, all marks are available for the remainder of the question | | **(3)** |
| **(b)(i)** <br> **Way 2** <br> $\mathbf{A}\mathbf{x}=2\mathbf{x}$ | $\mathbf{A}\mathbf{x}=2\mathbf{x}\Rightarrow$ <br> $ax+ay+z=2x$ <br> $-ax+4y=2y$ <br> $4x+ay+5z=2z$ | Uses $\mathbf{A}\mathbf{x}=2\mathbf{x}\;\left[or\,(\mathbf{A}-2\mathbf{I})\mathbf{x}=0\right]$ to obtain three simultaneous equations. Allow if given as two equal vectors. | M1 |
| | $\Rightarrow\; a^2+3a-10=0$ <br> $\Rightarrow\;(a-2)(a+5)=0\;\Rightarrow a=...$ | Forms and solves a 3TQ in *a*. Apply usual rules. If calculator used must obtain one correct solution for their 3TQ which could be complex. <br> Could be implied. <br> **Requires previous M mark.** | **d**M1 |
| | $(a>0\;\Rightarrow)\,a=2$ | Correct value of *a* from correct work. <br> If –5 is offered imply its rejection if 2 alone is used in (ii) | A1 |
| | If *a* = 2 is arrived at fortuitously, all marks are available for the remainder of the question | | **(3)** |

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| **5(b)(ii)** | $(2-\lambda)(4-\lambda)(5-\lambda)+4(5-\lambda)+(-4-16+4\lambda)=0$ <br> $\Rightarrow(5-\lambda)\left[(2-\lambda)(4-\lambda)+4-4\right]=0$ <br> $\Rightarrow(5-\lambda)(2-\lambda)(4-\lambda)=0\Rightarrow\lambda=...$ | Uses their value of $a$ in a recognisable attempt at a characteristic equation and achieves a real non-zero eigenvalue $\neq 2$. There must be some algebra but it may be poor. | M1 |
| | 4 and 5 | Both correct (no extra) and from correct work | A1 |
| | For information the cubic is $\pm\left(\lambda^3-11\lambda^2+38\lambda-40\right)=0$ | | **(2)** |
| **(c)** | $\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}="4"\begin{pmatrix}x\\y\\z\end{pmatrix}$ $or\left(\mathbf{A}-"4"\mathbf{I}\right)\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow$ $\begin{matrix}2x+2y+z=4x\\-2x+4y=4y\\4x+2y+5z=4z\end{matrix}$ $or$ $\begin{matrix}-2x+2y+z=0\\-2x=0\\4x+2y+z=0\end{matrix}$ <br> OR <br> $\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}="5"\begin{pmatrix}x\\y\\z\end{pmatrix}$ $or\left(\mathbf{A}-"5"\mathbf{I}\right)\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow$ $\begin{matrix}2x+2y+z=5x\\-2x+4y=5y\\4x+2y+5z=5z\end{matrix}$ $or$ $\begin{matrix}-3x+2y+z=0\\-2x-y=0\\4x+2y+z=0\end{matrix}$ <br> Uses $\mathbf{Ax}=\lambda\mathbf{x}$ $or$ $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$ with their value of $a$ and a real non-zero value of $\lambda\neq 2$ to obtain three simultaneous equations (allow if given as two equal vectors) <br> Alternatively attempts vector product of two rows of $\mathbf{A}-"4"\mathbf{I}$ | | M1 |
| | $\pm\begin{pmatrix}0\\-1\\2\end{pmatrix}$ **or** $\pm\begin{pmatrix}1\\-2\\7\end{pmatrix}$ | One correct eigenvector. As shown or multiple or with components multiplied by e.g. "*k*" Accept e.g., $x=0,\ y=-1,\ z=2$ | A1 |
| | $\pm\begin{pmatrix}0\\-1\\2\end{pmatrix}$ **and** $\pm\begin{pmatrix}1\\-2\\7\end{pmatrix}$ | Both correct eigenvectors. As shown or multiple or with components multiplied by e.g. *k* Accept $x=...,\ y=...,\ z=...$ Both these 2 A marks could be implied by their normalised eigenvectors | A1 |
| | $\pm\frac{1}{\sqrt{5}}\begin{pmatrix}0\\-1\\2\end{pmatrix},\ \pm\frac{1}{\sqrt{54}}\begin{pmatrix}1\\-2\\7\end{pmatrix}$ oe | M1: A correct method to normalise at least one of their eigenvectors A1: Both correct. Allow any exact equivalents. Isw | M1 A1 |
| | All marks available regardless of how $a=2,\ \lambda_2=4\ \&\ \lambda_3=5$ have been obtained | | **(5)** |
| | | | **Total 12** |

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| **6(a)** | $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\begin{cases}a(1-\cos\theta)\\ \text{or}\\ a-a\cos\theta\end{cases}$ **or** $\frac{\mathrm{d}y}{\mathrm{d}\theta}=a\sin\theta$ | At least **one** correct derivative | B1 |
| | $a^2(1-\cos\theta)^2+(a\sin\theta)^2$ $=a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$ $=2a^2(1-\cos\theta)$ | Squares and adds their derivatives and uses $\cos^2\theta+\sin^2\theta=1$ to obtain an expression in $\cos\theta$ only (not $\cos^2\theta$) Could be implied | M1 |
| | $=2a^2\left(1-\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)\right)=4a^2\sin^2\frac{\theta}{2}$ | **d**M1: Replaces $\cos\theta$ with $\pm1\ \pm2\sin^2\frac{\theta}{2}$ or equivalent trig work (sign errors only on identities) to obtain an expression in $\sin^2\frac{\theta}{2}$ only **Requires previous M mark.** Can be implied. A1: Achieves $4a^2\sin^2\frac{\theta}{2}$ or $k$ = 4 from correct work | **d**M1 A1 |
| | | | **(4)** |
| **(b)** | $\text{S.A.}=(2\pi)\int y\sqrt{\left\{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2\right\}}\mathrm{d}\theta$ $=(2\pi)\int_{(0)}^{(2\pi)}a(1-\cos\theta)\left(2a\sin\frac{\theta}{2}\right)\mathrm{d}\theta$ | Applies $y\sqrt{\left\{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2\right\}}$ with their $ka^2\sin^2\frac{\theta}{2}$ and square roots. The result of the square root may be incorrect but must be of the form $p\sin\frac{\theta}{2}$ Allow a slip replacing *y* but they must not have used *x*, $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ or $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ for *y* Allow the letter *k* or an invented value. $2\pi$ may be absent or wrong. Integral not required. | M1 |
| | $=(2\pi)2a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\cos\theta\right)\mathrm{d}\theta$ $\Rightarrow(2\pi)2a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)\right)\mathrm{d}\theta$ or e.g., $\Rightarrow(2\pi)2a^2\int_{(0)}^{(2\pi)}2\sin^3\frac{\theta}{2}\,\mathrm{d}\theta$ | Uses trig identity/identities (condoning sign errors) to obtain an expression with arguments of $\frac{\theta}{2}$ only. Allow the letter *k* or an invented value. $2\pi$ may be absent or wrong. Integral not required. **Dependent on previous M mark.** | **d**M1 |
| | Scheme continues... | | |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **6(b) cont.** | $\left(=(2\pi)4a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\mathrm{d}\theta\right)$ <br> $S=8\pi a^2\left[-2\cos\frac{\theta}{2}+\frac{2}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$ <br> or e.g., $\pi a^2\left[-16\cos\frac{\theta}{2}+\frac{16}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$ | A correct expression for the surface area ignoring limits ft their numerical *k*, i.e., $S=2k\pi a^2\left[-2\cos\frac{\theta}{2}+\frac{2}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$ oe <br> If they integrate in a piecemeal fashion, award this mark if they have a correct expression for their *k* when integration is completed – any partial evaluations must be correct for their *k* | A1ft |
| | $=8\pi a^2\left[\left(-2\cos\frac{2\pi}{2}+\frac{2}{3}\cos^3\frac{2\pi}{2}\right)-\left(-2\cos 0+\frac{2}{3}\cos^3 0\right)\right]$ | Substitutes correct limits and attempts to subtract either way round following a completed attempt at integration with a numerical *k*. **Requires previous M marks** and must have used $2\pi$ . <br> Look for evidence of correct limit substitution and subtraction. There may be slips but insist on limits being applied on all integrations if they have been carried out separately. Algebraic results of integration must be seen | **dd**M1 |
| | $=\frac{64}{3}\pi a^2$ | Correct exact answer. Accept equivalent fractions. | A1 |
| | All marks available regardless of how *k* = 4 was obtained | | **(5)** |
| | | | **Total 9** |
| | Other integration methods: <br> Allow the second M mark to be available before any attempt at integration is made. Otherwise the second M is only awarded if they **complete** integration without any loss of the required forms (i.e., sign and coefficient errors only and just sign errors only with any trig identities). The first A (ft) mark is for a fully correct expression ignoring limits for their *k*. The last two marks are the same as the main scheme. <br> For information: <br> Applying parts to $\int\sin\frac{\theta}{2}\cos\theta\ \mathrm{d}\theta$ gives $\frac{2}{3}\left(\cos\frac{\theta}{2}+2\sin\frac{\theta}{2}\right)$ <br> Using addition formulae: <br> $\int\sin\frac{\theta}{2}\cos\theta\ \mathrm{d}\theta=\frac{1}{2}\int\left(\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}\right)\mathrm{d}\theta=\frac{1}{2}\left(2\cos\frac{\theta}{2}-\frac{2}{3}\cos\frac{3\theta}{2}\right)$ | | |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **7(a)** | $\begin{pmatrix}0\\3\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}=8\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ | M1: Attempts vector product of two vectors in the plane. Unless there is a full clear method they must achieve two correct components<br>A1: $\pm(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ or multiple | M1 A1 |
| | Allow any vector notation throughout this question | | **(2)** |
| **(b)** | $l$ has direction vector $\pm(2\mathbf{j}+2\mathbf{k})$ | Correct direction for $l$ | B1 |
| | $(\cos\alpha \text{ or } \sin\theta=)$<br>$\left\|\frac{"(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})"."(2\mathbf{j}+2\mathbf{k})"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\|=\left\|\frac{"(8)(0)+(-2)(2)+(-3)(2)"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{0^2+2^2+2^2}"}\right\|\left(=\left\|\frac{-10}{\sqrt{77}\times\sqrt{8}}\right\| \text{ or } \left\|\frac{-5\sqrt{154}}{154}\right\|\right)$<br>M1: For the scalar product of their normal and direction vector divided by the product of the magnitudes of their vectors. The first expression above oe is sufficient. There must have been a valid attempt at both vectors. Allow copying errors/slips if intention is clear. Modulus not required.<br>A1ft: A correct ft numerical expression with scalar product calculated as shown by second expression or better. Allow a decimal correct to 2sf. Modulus not required. Ignore labelling. Actual decimal is 0.40291148...<br>Implied by awrt 24 or 66 or 114 provided some work and nothing incorrect seen.<br>Allow awrt 0.41, 1.16 or 1.99 if working in radians. | | M1 A1ft |
| | Acute angle between $l$ and $P$<br>$=90-\alpha=90-66.23968409...$<br>or $\theta=23.76031591...\Rightarrow 24°$<br>to the nearest degree | awrt 24 from correct work which could be minimal. Degrees symbol not required.<br>**Mark final answer.** | A1 |
| | | | **(4)** |
| | Note that a vector product could be used:<br>M1: $\left\|\frac{\left\|"(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})"\times"(2\mathbf{j}+2\mathbf{k})"\right\|}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\|$ A1: $\left\|\frac{"\sqrt{2^2+16^2+16^2}"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\|$ $\left(=\frac{2\sqrt{129}}{\sqrt{77}\sqrt{8}}=0.9152389511...\right)$<br>The modulus of the numerator is required for any marks | | |
| **(c)**<br>**Way 1**<br>**Parallel planes** | $(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).("8\mathbf{i}-2\mathbf{j}-3\mathbf{k}")=-5$<br>or<br>$(6\mathbf{i}-3\mathbf{j}-6\mathbf{k}).("8\mathbf{i}-2\mathbf{j}-3\mathbf{k}")=72$ | M1: Finds a value for the scalar product of a position vector of a point in the plane or the given point and their normal.<br>A1: –5 or 72 (or 5 or -72 if normal is in the opposite direction). May be seen as a distance e.g., $\frac{-5}{\sqrt{"77"}}$ | M1 A1 |
| | Shortest distance is<br>$\left\|\frac{-5-72}{\sqrt{77}}\right\|=\frac{77}{\sqrt{77}}$ or $\sqrt{77}$ | **d**M1: Having attempted both scalar products, obtains a numerical expression for the distance.<br>Award for $\frac{\pm"5"\pm"72"}{\sqrt{"8"^2+"2"^2+"3"^2}}$<br>**Dependent on previous M mark.**<br>A1: Correct exact distance. Isw | **d**M1 A1 |
| | | | **(4)** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **7(c)**<br><br>**Way 2**<br><br>**Perp. distance formula** | $(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).(\text{"}8\mathbf{i}-2\mathbf{j}-3\mathbf{k}\text{"})=-5$ | M1: Finds a value for the scalar product of a position vector to a point the plane and their normal.<br>A1: -5 (or 5 if normal is in the opposite direction) | M1 A1 |
| | "$8x-2y-3z+5=0$"<br>Shortest distance is<br>$\frac{\|(\text{"}8\text{"})(6)+(\text{"}-2\text{"})(-3)+(\text{"}-3\text{"})(-6)+\text{"}5\text{"}\|}{\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}}$<br>$=\frac{77}{\sqrt{77}}$ or $\sqrt{77}$ | **d**M1: Uses distance formula with their normal and plane equation to reach a numerical expression for the distance. Condone sign slip on their -5 and their *d* must not be zero.<br>**Dependent on previous M mark.**<br>A1: Correct exact distance. Isw | **d**M1 A1 |
| | | | **(4)** |
| **Way 3**<br><br>**Projection /resolving formula** | Let *Q* be the point on the plane (1, 2, 3)<br>then $\overrightarrow{PQ}=(\mathbf{i}+2\mathbf{j}+3\mathbf{k})-(6\mathbf{i}-3\mathbf{j}-6\mathbf{k})$<br>$=-5\mathbf{i}+5\mathbf{j}+9\mathbf{k}$ | M1: Attempts vector from given point to a point on the plane<br>A1: Correct vector $(\pm)$ | M1 A1 |
| | Shortest distance is $\|\overrightarrow{PQ}.\mathbf{n}\|=$<br>$\frac{\|(\text{"}-5\mathbf{i}+5\mathbf{j}+9\mathbf{k}\text{"}).(\text{"}8\mathbf{i}+-2\mathbf{j}+-3\mathbf{k}\text{"})\|}{\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}}=...$<br>$=\frac{77}{\sqrt{77}}$ *or* $\sqrt{77}$ | **d**M1: Uses formula with their vectors to reach a numerical expression for the distance<br>**Dependent on previous M mark.**<br>A1: Correct exact distance. Isw | **d**M1 A1 |
| | | | **(4)** |
| **Way 4**<br><br>**Example of method involving the point where the line meets plane** | Line through given point in direction of normal is<br>$r=(6\mathbf{i}-3\mathbf{j}-6\mathbf{k})+\lambda(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})$<br>& meets plane "$8x-2y-3z+5=0$" when<br>$8(6+8\lambda)-2(-3-2\lambda)-3(-6-3\lambda)+5=0$<br>$\Rightarrow\lambda=-1$ | M1: Uses line through given point in the direction of their normal and substitutes into their plane to find a value for $\lambda$. The *d* in their plane equation must not be zero<br>A1: Correct value | M1 A1 |
| | $\|-1(\text{"}8\mathbf{i}+-2\mathbf{j}+-3\mathbf{k}\text{"})\|=\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}$<br>Or point of intersection is<br>(6 –"8", –3 –"–2", –6 –"–3")<br>= (–2, –1, –3) and distance is<br>$\sqrt{(6-\text{"}-2\text{"})^2+(-3-\text{"}-1\text{"})^2+(-6-\text{"}-3\text{"})^2}$<br>$\Rightarrow\sqrt{77}$ | **d**M1: Attempts $\|\lambda\mathbf{n}\|$ or finds point on the plane and obtains numerical expression for distance between this point and the given point<br>**Dependent on previous M mark.**<br>A1: Correct exact distance. Isw | **d**M1 A1 |
| | | | **(4)** |
| | Marks are scored through the ay which is the best overall match for the attempt.<br>Credit for work done in (b) is only available for part (c) if it is used in part (c). | | |
| | | | **Total 10** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **8(a)** **Way 1** | $I_n=\int\cos^n x\,\mathrm{d}x=\int\cos x\cos^{n-1}x\,(\mathrm{d}x)$ | Correct split. Could be implied by their work | M1 |
| | $=\sin x\cos^{n-1}x+\int(n-1)\cos^{n-2}x\sin^2 x\,(\mathrm{d}x)$ | Obtains $p\sin x\cos^{n-1}x+\int q\cos^{n-2}x\sin^2 x\,(\mathrm{d}x)$ oe **Requires previous M mark.** | **d**M1 |
| | $=\sin x\cos^{n-1}x+\int(n-1)\cos^{n-2}x\left(1-\cos^2 x\right)(\mathrm{d}x)$ | Replaces $\sin^2 x$ with $1-\cos^2 x$ to achieve a correct expression for $I_n$ | A1 |
| | $=\sin x\cos^{n-1}x+(n-1)I_{n-2}-(n-1)I_n$ <br> $\Rightarrow I_n=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}I_{n-2}$ * | Proceeds to the given answer with at least one intermediate step and no errors. Condone missing "d$x$"s but there must be no missing arguments. Any clear bracketing error must be recovered **before** given answer. | A1* |
| | | | **(4)** |
| **Way 2** | $I_n=\int\cos^n x\,\mathrm{d}x=\int\cos^2 x\cos^{n-2}x\,(\mathrm{d}x)$ <br> $=\int\left(1-\sin^2 x\right)\cos^{n-2}x\,(\mathrm{d}x)$ | Correct split **and** replaces $\cos^2 x$ with $1-\sin^2 x$ | M1 |
| | $=\int\left(\cos^{n-2}x-\cos^{n-2}x\sin^2 x\right)(\mathrm{d}x)$ <br> $=\int\cos^{n-2}x\,(\mathrm{d}x)-\int\left(\sin x\sin x\cos^{n-2}x\right)(\mathrm{d}x)=\ldots$ <br> M1: Expands, splits and obtains $p\int\cos^{n-2}x\,(\mathrm{d}x)+q\cos^{n-1}x\sin x+\int r\cos^n x\,(\mathrm{d}x)$ oe <br> **Requires previous M mark.** <br> A1: Correct expression for $I_n$: $\int\cos^{n-2}x\,(\mathrm{d}x)-\left(-\frac{1}{n-1}\cos^{n-1}x\sin x+\int\frac{1}{n-1}\cos^n x\,(\mathrm{d}x)\right)$ oe | | **d**M1 A1 |
| | $=I_{n-2}+\frac{1}{n-1}\cos^{n-1}x\sin x-\frac{1}{n-1}I_n$ <br> $\Rightarrow I_n=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}I_{n-2}$ * | Proceeds to the given answer with at least one intermediate step and no errors. Condone missing "d$x$"s but there must be no missing arguments. Any bracketing error must be recovered **before** given answer. | A1* |
| | | | **(4)** |
| **(b)** | $I_n=\frac{1}{n}\left[\cos^{n-1}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{n-1}{n}I_{n-2}$ or $=\frac{1}{n}(n-1)I_{n-2}$ <br> $I_2=\frac{1}{2}\left[\cos^{2-1}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{2-1}{2}I_0$ or $=\frac{1}{2}I_0$ | Uses the RF to obtain an expression for $I_n$ in terms of $I_{n-2}$ or $I_2$ in terms of $I_0$ <br> Condone if necessary if limits are absent. | M1 |
| | $I_n=\frac{(n-1)(n-3)\ldots5\times3\times1}{n(n-2)(n-4)\ldots6\times4\times2}I_0$ <br> with dots & at least 3 terms in each product (first 2 & last, or first & last 2) | Correct expression for $I_n$ in terms of $I_0$ oe following correct work including 2 applications of the reduction formula (which could be embedded) **prior** to this answer. $I_0$ may have been calculated previously but do not allow just the final printed answer to imply this mark. | A1 |
| | e.g., $I_0=\int_0^{\frac{\pi}{2}}\mathrm{d}x=\frac{\pi}{2}$ or $I_0=\left[x\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$ or $I_0=\frac{\pi}{2}-0$ | Correct value for $I_0$ - requires written evidence of integration (minimal) | B1 |
| | $\therefore I_n=\frac{(n-1)(n-3)\ldots5\times3\times1}{n(n-2)(n-4)\ldots6\times4\times2}\times\frac{\pi}{2}$ * <br> Allow extra terms in both products. | Proceeds to given answer. Requires all previous marks. Withhold this mark if no $\frac{1}{k}\left[\cos^{k-1}x\sin x\right]_0^{\frac{\pi}{2}}$ is seen or expression just disappears – one such expression must be replaced by "0" or have substitution seen | A1* |
| | Attempts via proof by induction will be reviewed. | | **(4)** |
| | Attempts may be seen via $I_n=\frac{(n-1)(n-3)\ldots3}{n(n-2)\ldots4}I_2$ and $I_2=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}=\frac{1}{2}\times\frac{\pi}{2}$ | | |

| Question Number | Scheme | Notes | Marks |
| --- | --- | --- | --- |
| **8(c)** | $\int_0^{\frac{\pi}{2}} \cos^6 x \sin^2 x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^6 x\left(1-\cos^2 x\right)\mathrm{d}x$ | Replaces $\sin^2 x$ with $1-\cos^2 x$<br>Can be implied by an attempt at $I_6 - I_8$ | M1 |
| | $= I_6 - I_8 = \left(\frac{5\times3\times1}{6\times4\times2} - \frac{7\times5\times3\times1}{8\times6\times4\times2}\right)\frac{\pi}{2}$ | Any correct numerical expression for the integral | A1 |
| | $\left(= \frac{5}{32}\pi - \frac{35}{256}\pi =\right)\frac{5}{256}\pi$ oe | Correct exact value. Accept equivalent fractions and allow e.g., $\left(\frac{5}{128}\right)\frac{\pi}{2}$ | A1 |
| | **This is a "Hence" and requires clear use of $I_6 - I_8$**<br>For the A marks there must be no evidence that the answer has been arrived at without using part (b). There is no credit in (b) for work in (c).<br>Just "$I = \frac{5}{256}\pi$" is 0/3 but just "$I_6 - I_8 = \frac{5}{256}\pi$" is 3/3 | | |
| | | | **(3)** |
| | | | **Total 11** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **9(a)(i)** | $b^2 = a^2(1-e^2) \Rightarrow 1 = 9(1-e^2)$<br>$\Rightarrow e^2 = ...\left(\frac{8}{9}\right),\ e = \frac{2\sqrt{2}}{3}$ or $\frac{\sqrt{8}}{3}$ | M1: Uses a correct eccentricity formula with correct values for *a* and *b* and obtains a value for $e^2$ or $e$<br>A1: Correct value for *e* (not $\pm$)<br>Could be implied | M1 A1 |
| | Foci are $(\pm 2\sqrt{2}, 0)$ or $(\pm\sqrt{8}, 0)$ | B1: Both correct foci as coordinates<br>Condone any use of a negative *e*<br>**Note that this is not an ft mark.** | B1 |
| | | | **(3)** |
| **(a)(ii)** | $x = \pm\frac{9\sqrt{2}}{4}$ or $\pm\frac{9\sqrt{8}}{8}$ or $\pm\frac{9}{\sqrt{8}}$ oe<br>Both correct **equations**. Requires single fraction.<br>**Allow ft**: $x = \pm\frac{\mathbf{3}}{\text{their}\, e}$ computed into a single fraction, condoning e < 0<br>Allow "$x_1 = ..., x_2 = ...$"<br>Condone, e.g., "$x = \pm\frac{a}{e} = \frac{9\sqrt{2}}{4}$ or $-\frac{9\sqrt{2}}{4}$" but just "$\frac{a}{e} = \pm\frac{9\sqrt{2}}{4}$" is B0 | | B1ft |
| | | | **(2)** |
| **(b)**<br><br>**Way 1**<br><br>$PF = ePM$ | $\lvert PF_1\rvert = e\lvert PM_1\rvert$ or $\lvert PF_2\rvert = e\lvert PM_2\rvert$ oe | States this definition of an ellipse. | M1 |
| | $\lvert PF_1\rvert + \lvert PF_2\rvert = \underline{e(\lvert PM_1\rvert + \lvert PM_2\rvert)}$ or $\underline{e(\lvert M_1M_2\rvert)}$<br>"$\frac{2\sqrt{2}}{3}$"$\times 2 \times$"$\frac{9\sqrt{2}}{4}$" oe<br>or $\lvert PF_1\rvert + \lvert PF_2\rvert =$<br>$=$"$\frac{2\sqrt{2}}{3}$"$\left(\text{"}\frac{9\sqrt{2}}{4}\text{"} - x\right) +$"$\frac{2\sqrt{2}}{3}$"$\left(\text{"}\frac{9\sqrt{2}}{4}\text{"} + x\right)$ | Correct method for a numerical expression (or with cancelling "$x$"s) for $\lvert PF_1\rvert + \lvert PF_2\rvert$ with their *e* and directrix.<br>**One of the underlined expressions must be seen for the first approach.**<br>**Requires previous M mark.** | **d**M1 |
| | $= 6$ * | Fully correct proof. Modulus signs are not required. | A1* |
| **Way 1**<br><br>**Guidance** | If they work in *a* and *e*, $e \times 2 \times \frac{a}{e}$ is only acceptable if $\underline{e(\lvert PM_1\rvert + \lvert PM_2\rvert)}$ or $\underline{e(\lvert M_1M_2\rvert)}$ is seen (as with using the values) and $e\left(\frac{a}{e} - x\right) + e\left(\frac{a}{e} + x\right)\ (\Rightarrow 2a)$ is acceptable but note in both these general cases the second M mark becomes available when *a* = 3 is substituted.<br>**<u>The second M is not available for any work which relies on</u>** $\lvert PF_1\rvert = \lvert PF_2\rvert$<br>**<u>Their proof needs to be shown to be valid for any position of *P*</u>**<br>So $\lvert PF_1\rvert + \lvert PF_2\rvert = \frac{2\sqrt{2}}{3} \times \frac{9\sqrt{2}}{4} + \frac{2\sqrt{2}}{3} \times \frac{9\sqrt{2}}{4}$ or using $e \times \frac{a}{e} + e \times \frac{a}{e}$ cannot score the second M without $\underline{e(\lvert PM_1\rvert + \lvert PM_2\rvert)}$ or $\underline{e(\lvert M_1M_2\rvert)}$ being seen.<br>If *e* appears as a value it must be correct for the final mark.<br>Just $\lvert PF_1\rvert + \lvert PF_2\rvert = 2a = 2 \times 3 = 6$ is 0/3<br>Having earned the first mark in Way 1, some candidates proceed to work with a specific point on the ellipse as in Way 2. Further credit is only available if they clearly state e.g, "$\lvert PF_1\rvert + \lvert PF_2\rvert$ is constant for any *P*" | | **(3)** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **9(b)** **Way 2** $PF_1 + PF_2 = k$ | $\lvert PF_1\rvert + \lvert PF_2\rvert = \lvert QF_1\rvert + \lvert QF_2\rvert$ where *P* and *Q* are any points on the ellipse oe | States this oe definition of an ellipse, justified by explanation. Accept e.g., "$\lvert PF_1\rvert + \lvert PF_2\rvert$ is constant for any *P*" | M1 |
| | e.g. *Q* is where *E* crosses positive *x*-axis $\Rightarrow \lvert PF_1\rvert + \lvert PF_2\rvert = 3 - "2\sqrt{2}" + 3 + "2\sqrt{2}"$ <br> *Q* is where *E* crosses positive *y*-axis $\Rightarrow \lvert PF_1\rvert + \lvert PF_2\rvert = 2\sqrt{1^2 + "2\sqrt{2}"^2}$ <br> *Q* is on *E* directly above $F_1$ $\Rightarrow \lvert PF_1\rvert + \lvert PF_2\rvert = \sqrt{1 - \frac{("2\sqrt{2}"^2)}{9}} + \sqrt{(2\times"2\sqrt{2}")^2 + 1 - \frac{("2\sqrt{2}"^2)}{9}}$ | Correct method for a numerical value for $\lvert PF_1\rvert + \lvert PF_2\rvert$ using another point on the ellipse and their foci. **Requires previous M mark.** | **d**M1 |
| | $= 6$ * | Fully correct proof. Modulus signs are not required. | A1* |
| | | | **(3)** |
| **Way 3** **Point in terms of $\theta$** | $P(3\cos\theta, \sin\theta)$ <br> $\lvert PF_1\rvert^2 = (3\cos\theta - "2\sqrt{2}")^2 + \sin^2\theta$ <br> or $\lvert PF_2\rvert^2 = (3\cos\theta + "2\sqrt{2}")^2 + \sin^2\theta$ | Correct general point in parametric form and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of *a*, *b* and $\theta$ | M1 |
| | $\lvert PF_1\rvert + \lvert PF_2\rvert =$ $\sqrt{8\cos^2\theta - 12\sqrt{2}\cos\theta + 9} + \sqrt{8\cos^2\theta + 12\sqrt{2}\cos\theta + 9}$ | Correct method for $\lvert PF_1\rvert + \lvert PF_2\rvert$ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when *a* and *b* are substituted. **Requires previous M mark.** | **d**M1 |
| | $\lvert PF_1\rvert + \lvert PF_2\rvert =$ $3 - 2\sqrt{2}\cos\theta + 3 + 2\sqrt{2}\cos\theta = 6$* | Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way. | A1* |
| | | | **(3)** |
| **Way 4** **Point in terms of *x*** | $P\left(x, \sqrt{1 - \frac{x^2}{9}}\right)$ or $P\left(x, \sqrt{\frac{9 - x^2}{9}}\right)$ <br> $\lvert PF_1\rvert^2 = ("2\sqrt{2}" - x)^2 + 1 - \frac{x^2}{9}$ <br> or $\lvert PF_2\rvert^2 = (x + "2\sqrt{2}")^2 + 1 - \frac{x^2}{9}$ | Correct general point in terms of *x* and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of *a*, *b* and *x*. | M1 |
| | $\lvert PF_1\rvert + \lvert PF_2\rvert = \sqrt{\frac{8}{9}x^2 - 4\sqrt{2}x + 9} + \sqrt{\frac{8}{9}x^2 + 4\sqrt{2}x + 9}$ | Correct method for $\lvert PF_1\rvert + \lvert PF_2\rvert$ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when *a* and *b* are substituted. **Requires previous M mark.** | **d**M1 |
| | $\lvert PF_1\rvert + \lvert PF_2\rvert = 3 - \frac{2\sqrt{2}}{3}x + 3 + \frac{2\sqrt{2}}{3}x = 6$* | Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way. | A1* |
| | Creditworthy alternative approaches will be reviewed | | **(3)** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **9(c)** | $x^2+9(2x+c)^2=9$ or $\frac{x^2}{9}+(2x+c)^2=1$ | Substitutes line into the ellipse equation. Condone slips provided intention clear. | M1 |
| | $37x^2+36cx+9c^2-9=0$ <br> or e.g., $\frac{37}{9}x^2+4cx+c^2-1=0$ | Correct quadratic with $x^2$ terms collected (could be implied) | A1 |
| | ½ (sum of roots) $\Rightarrow$ $(x=)\frac{-18c}{37}$ <br> or <br> $(x=)\frac{1}{2}\left(\frac{-36c+\sqrt{(36c)^2-4(37)(9c^2-9)}}{2(37)}+\frac{-36c-\sqrt{(36c)^2-4(37)(9c^2-9)}}{2(37)}\right)$ <br> M1: Correct attempt at ½ (sum of roots), i.e., $-\frac{b}{2a}$ for their quadratic. <br> Ignore how the expression is labelled. <br> **Requires previous M mark.** <br> A1: Any correct **equation** in $x$ and $c$ <br> Allow this mark if e.g., $x$ is seen as $M_x$ | | **d**M1 A1 |
| | $\Rightarrow c=\text{"}-\frac{37}{18}\text{"}x\Rightarrow y=2x+\left(\text{"}-\frac{37}{18}\text{"}\right)x$ <br> or $x=\text{"}-\frac{18}{37}\text{"}c\Rightarrow y=2\times\text{"}-\frac{18}{37}\text{"}c+c\Rightarrow ...\left(y=\frac{c}{37}\Rightarrow\frac{y}{x}=-\frac{1}{18}\right)$ | Substitutes their $c = px$ into the line to obtain an equation in ***x* and *y*** only. <br> Allow e.g., $x_M$ and $y_M$ and condone e.g., suffixes of $P$ & $Q$ <br> This may also be achieved by e.g., finding $y$ in terms of $c$ and then eliminating $c$ with their equation in $x$ and $c$ <br> Must not be using "$M_x$" or "$M_y$" etc. but imply this mark from a locus equation in $x$ and $y$ or $x_{...}$ and $y_{...}$ with appropriate suffixes <br> **Requires both previous M marks** | **dd**M1 |
| | $\Rightarrow y_{...}=-\frac{1}{18}x_{...}$ oe <br> $\therefore$ $l$ passes through the origin oe * | Obtains correct equation for locus (accept equivalents) **and** makes conclusion e.g., "passes/goes through origin/$O$/(0,0)" but allow "shown"/"as required"/"QED" etc. <br> **Requires all previous marks.** | A1* |
| | | | **(6)** |
| | | | **Total 13** |
| | | | **PAPER TOTAL: 75** |